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Theory of interacting Josephson junctions (Josephson lattices)

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Abstract. A theoretical approach is derived to describe a Josephson junction of complex form with self-crossings. Results of the approach are used to study properties of granular superconducting systems with Josephson junctions between grains (Josephson networks or lattices). We show that properties of Josephson lattices depend on the grain size (L). If L is smaller than the Josephson length (δ), then the lattices are described by the energy functional of the XY model in which internal magnetic fields are taken into account self-consistently. On the basis of the functional we find the low critical field (H^*_{c1}) and the pinning energy of vortices. We study also a case of high magnetic fields ($H \gg H^*_{c1}$). In this case the lattices are also described by the XY model but the coupling strength between grains may depend strongly on H.

1. Introduction

A large number of experimental and theoretical papers have been devoted to Josephson networks with superconducting grains coupled by Josephson junctions (see, for example, NATO Workshop 1988, Clem 1988, and many others). Different types of Josephson networks may be fabricated using modern lithographic techniques. On the other hand, an extensive investigation of high-T_c ceramic superconductors gives a further impetus to a more profound study of Josephson networks. Interaction between Josephson junctions has an important effect on properties of the networks. The interaction is mediated by magnetic fields and takes place either between junctions lying at a distance of the order of the London penetration depth λ, or between crossed junctions.

In this paper we develop a general theory of a Josephson junction that has a complicated form including self-crossings (section 2). In section 3 we apply the theory to describe systems of parallel Josephson junctions. Section 4 is devoted to an investigation of two crossed Josephson junctions. We consider rectangular Josephson lattices in terms of the effective medium (section 5). The assumptions used in sections 3 and 4 are discussed in the appendix, where we obtain an exact solution for the problem of magnetic field penetration in an isolated linear Josephson junction. In section 6 we point out that our approach may be generalised to study the interaction between Josephson junctions and Abrikosov vortices.

It should be noted that we will study only Josephson networks that are uniform along the z axis. In this case the problem under consideration becomes two-dimensional in fact, and three-dimensional effects are ignored ((see, for example, papers of Dasgupta

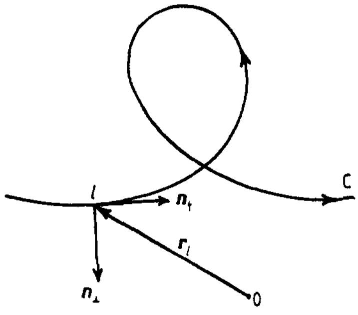

Figure 1. The curve C represents a Josephson junction, that is a thin dielectric layer, for example, with a self-crossing. The variable l is a continuous variable along the curve C; and $\boldsymbol{n}_t$ and $\boldsymbol{n}_\perp$ are tangential and normal unit vectors, respectively.

and Halperin (1981) and Bartholomew (1983), in which some of these 3D effects have been studied).

2. Josephson junctions of complicated form

Let us consider a Josephson junction that in the xy plane is represented by an arbitrary curve (for example see figure 1). Along the z axis the system under consideration is supposed to be uniform. The Josephson junction whose thickness is small compared to the London penetration depth λ cuts the superconductor into grains. We assume that inside the grains the superconducting order parameter does not depend on coordinates. Magnetic field obeys the following equations:

$$\operatorname{rot} \boldsymbol{H} = (4\pi/c)\boldsymbol{j} \tag{1}$$

$$\frac{4\pi}{c}\boldsymbol{j} = \frac{1}{\lambda^2}\left(\frac{\phi_0}{2\pi}\nabla\varphi - \boldsymbol{A}\right) \tag{2}$$

where $\phi_0 = \pi\hbar c/e$ is the flux quantum and $\boldsymbol{A}$ is the vector potential. Substituting (2) into (1) we obtain the London equation

$$\Delta \boldsymbol{H} - \lambda^{-2}\boldsymbol{H} = 0 \tag{3}$$

which is correct everywhere except on junction surfaces. Below, we will suppose that the magnetic field $\boldsymbol{H}$ is parallel to the z axis and depends only on the variables x and y, i.e. $\boldsymbol{H} = (0, 0, H(x, y))$. In this case Δ is the two-dimensional Laplacian.

Now we consider boundary conditions on the junction surface. Let $H(x, y)$ be a continuous function whose normal derivative breaks on the junction surfaces. To satisfy the boundary condition we can introduce a 'surface charge' $\sigma(x, y)$ distributed on the junction surface. In this case a solution of the two-dimensional equation (3) may be written in the form

$$H(\boldsymbol{r}) = \int \mathrm{d}l\, \sigma(\boldsymbol{r}_l) K_0\left(\left|\frac{\boldsymbol{r} - \boldsymbol{r}_l}{\lambda}\right|\right) \tag{4}$$

where $\boldsymbol{r} = (x, y)$, $K_0(x)$ is the modified Bessel function and $\boldsymbol{r}_l$ is a radius vector of a point l on the junction (see figure 1). The integration in (4) is taken over the junction and l is a continuous variable along the junction. At each point l of the junction we define the normal unit vector $\boldsymbol{n}_\perp$ and the tangential unit vector $\boldsymbol{n}_t$ ($[\boldsymbol{n}_\perp \times \boldsymbol{n}_t] = \boldsymbol{n}_z$, where $\boldsymbol{n}_z$ is the unit vector along the z axis). Equation (4) does not take into account an external boundary of the considered system. Let the magnetic field $H(x, y)$ on the boundary be equal to an external magnetic field H_0. Then determination of $H(x, y)$ from equation (3) with this boundary condition is equivalent to determination of an electric potential in a metal, where the boundary condition may be taken into account by use of forces of the mirror reflection. If the boundary is flat then we obtain

$$H(r) = H_0 \exp(-x/\lambda) + \int \mathrm{d}l\, \sigma(r_l)\left[K_0\left(\frac{|r - r_l|}{\lambda}\right) - K_0\left(\frac{|r + r_l|}{\lambda}\right)\right] \tag{5}$$

where the boundary plane goes along the y axis and the superconductor occupies the region $x \geqslant 0$. At a distance $x \gg \lambda$ the effect of the boundary is negligible and equation (5) comes to equation (4).

To find $\sigma(r_l)$ we determine from (1) the normal and tangential components of the current j near the junction:

$$(j, n_\perp) = \frac{c}{4\pi}([n_z \times n_\perp], \nabla H) = \frac{c}{4\pi}(n_t, \nabla H) \tag{6}$$

$$(j, n_t) = \frac{c}{4\pi}([n_z \times n_t], \nabla H) = -\frac{c}{4\pi}(n_\perp, \nabla H). \tag{7}$$

According to (7) at a point l the jump of the tangential component of the current is

$$(j^+ - j^-, n_t) = -\frac{c}{4\pi}(n_\perp, \nabla H^+ - \nabla H^-) = \tfrac{1}{2}c\sigma(r_l). \tag{8}$$

In (8) the second equality is a consequence of (4). On the other hand, equation (2) gives

$$(j^+ - j^-, n_t) = \frac{c\phi_0}{8\pi^2\lambda^2}(\nabla\theta, n_t) \equiv \frac{c\phi_0}{8\pi^2\lambda^2}\frac{\partial\theta}{\partial l} \tag{9}$$

where

$$\nabla\theta = \nabla(\varphi^+ - \varphi^-) - \frac{2\pi}{\phi_0}(A^+ - A^-).$$

Here $\varphi^\pm$ and $A^\pm$ are values of φ and A on the two sides of the junction (see figure 1). From (8) and (9) it follows that $\sigma(l) \equiv \sigma(r_l)$ is equal to

$$\sigma(l) = \frac{\phi_0}{4\pi^2\lambda^2}\frac{\partial\theta}{\partial l}. \tag{10}$$

Inserting (9) into (4) we obtain a relation between the phase difference θ on the junction and the magnetic field $H(r)$ at a point $r \equiv (x, y)$:

$$H(r) = \frac{\phi_0}{4\pi^2\lambda^2}\int \mathrm{d}l\,\frac{\partial\theta}{\partial l}K_0\left(\frac{|r - r_l|}{\lambda}\right). \tag{11}$$

The normal component of the current (6) through the junction is equal to the Josephson current ($j_c \sin\theta$). Therefore using (6) and (11) we obtain

$$\sin\theta(l) = \frac{c}{4\pi j_c}\frac{\partial}{\partial l}H(r_l) = \frac{\delta^2}{\pi\lambda}\frac{\partial}{\partial l}\int \mathrm{d}l'\,\frac{\partial\theta(l')}{\partial l'}K_0\left(\frac{|r_l - r_{l'}|}{\lambda}\right) \tag{12}$$

where $\delta^2 = \hbar c^2/16\pi j_c e\lambda$ is the square of the Josephson length (Josephson 1965).

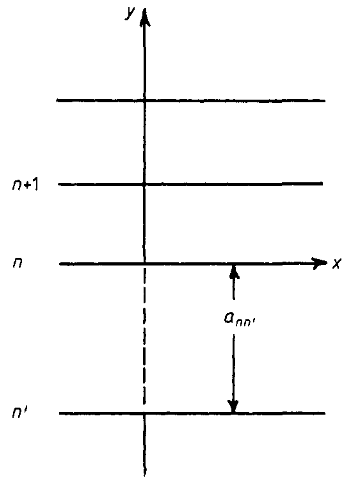

Figure 2. The system of parallel Josephson junctions (full lines); $a_{nn'}$ is the distance from junction with index n to junction n'.

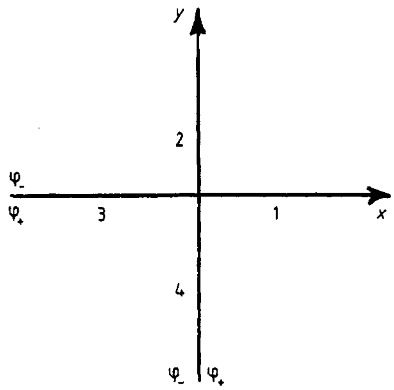

Figure 3. The simplest system consisting of two Josephson junctions that cross each other (full lines). One junction is along the x axis and the other is along the y axis. The rest of the volume is filled by a superconductor.

Let us suppose that the curve that represents the junction satisfies the following topological conditions: (i) there are no breaks and self-crossings; (ii) the minimal radius of the space curvature of the curve is larger than λ; (iii) if $|l_A - l_B| \gg \lambda$ for arbitrary points A and B lying on the junction, then $|r_A - r_B| \gg \lambda$ also; and (iv) for $\delta \gg \lambda$ the function $\partial\theta(l)/\partial l$ changes slowly over a distance of the order of λ. In this case we can take $\partial\theta(l')/\partial l'$ out of the integral (12) at $l = l'$. Therefore we have

$$\sin\theta(l) = \delta^2\,\partial^2\theta/\partial l^2. \tag{13}$$

This is the well known sine–Gordon equation obtained by Ferrel and Prange (1963).

In the next sections we will apply general equation (12) to study Josephson networks.

3. Systems of parallel junctions

Now we consider a system consisting of interacting Josephson junctions that are parallel to the x axis. In this case equation (12) takes the form

$$\sin\theta_n(x) = \frac{\delta^2}{\pi\lambda}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n'}\int \mathrm{d}x'\,\frac{\mathrm{d}\theta_{n'}(x')}{\mathrm{d}x'}\,K_0(\lambda^{-1}[a_{nn'}^2 + (x-x')^2]^{1/2}) \tag{14}$$

where $\theta_n(x)$ is the phase difference on the junction with index n' and $a_{nn'}$ is the distance along the y axis between junctions n and n' (figure 2). If $\mathrm{d}\theta_n(x)/\mathrm{d}x$ change slowly over a distance of the order of λ, then equation (14) leads to

$$\sin\theta_n = \delta^2\sum_{n'}\exp\left(\frac{-a_{nn'}}{\lambda}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\theta_{n'}(x). \tag{15}$$

In terms of the nearest-neighbour approach, equation (15) comes to the equation obtained by Volkov (1987). From equation (15) we can find the magnetic field penetration depth along the x axis. Let the superconducting system occupy the region $x \geqslant 0$ and $a_{nn'} = |n - n'|a$. Using the linearisation $\sin\theta_n \simeq \theta_n$ and supposing $\theta_n(x=0) = \theta_0$ at all n, we obtain

$$\theta_n(x) = \theta_0 \exp(-x/\delta^*)$$

where

$$\delta^* = \delta[\coth(a/2\lambda)]^{1/2} \tag{16}$$

is the effective penetration depth of the magnetic field. If $a \gg \lambda$ then the interaction between junctions is negligible and $\delta^* \simeq \delta$. In the contrary case $a \ll \lambda$ the interaction leads to increasing δ^* ($\delta^* = \delta(2\lambda/a)^{1/2} \gg \delta$). We can assume that, the larger δ^*, the smaller is the low critical magnetic field H^*_{c1}.

4. Interception of junctions and boundary conditions for Josephson lattices

Interception of Josephson junctions is an important case of interaction between junctions. First we discuss the interception of two linear junctions (figure 3). It is obvious that at distances larger than λ from the interception of the junctions we can neglect the interaction between the junctions. In this case the phase difference $\theta(l)$ satisfies equation (13), where $l = x, y$. To find a boundary condition at the interception of the junctions we integrate equation (11) over a small area (S) that contains the interception. If the flux of magnetic field through the area tends to zero in the limit $S \to 0$, then we obtain

$$\lim_{S\to 0} \int \mathrm{d}S\,\mathrm{d}l \frac{\partial\theta}{\partial l} K_0\left(\frac{|r - r_l|}{\lambda}\right) = 0. \tag{17}$$

Let us take into account the equality

$$\lambda^{-2} K_0\left(\frac{|r - r_l|}{\lambda}\right) = \Delta K_0\left(\frac{|r - r_l|}{\lambda}\right) - \delta(r - r_l)$$

and substitute it into (17). Since the integral of the first term (ΔK_0) is equal to zero, equation (17) takes the form

$$\theta^{(1)} + \theta^{(2)} = \theta^{(3)} + \theta^{(4)} \tag{18}$$

where $\theta^{(i)}, i = 1,2,3,4$, are limit values of $\theta(l)$ on rays with index i near the interception (see figure 3). For example, in accordance with figure 3 we have $\theta^{(1)} \equiv \theta(x \to +0)$, $\theta^{(2)} \equiv \theta(y \to +0)$, $\theta^{(3)} \equiv \theta(x \to -0)$ and $\theta^{(4)} \equiv \theta(y \to -0)$. It is important to note that equation (18) is exact and allows a discontinuity of the function $\theta(l)$ at the interception. We can use relation (18) as the boundary condition for equation (13) if the function $\theta(l)$ changes slowly over a distance of the order of λ from the interception. Although we cannot prove the suggestion exactly for the system considered above, in the appendix we give a proof of the suggestion in the related case of magnetic field penetration into a linear Josephson junction.

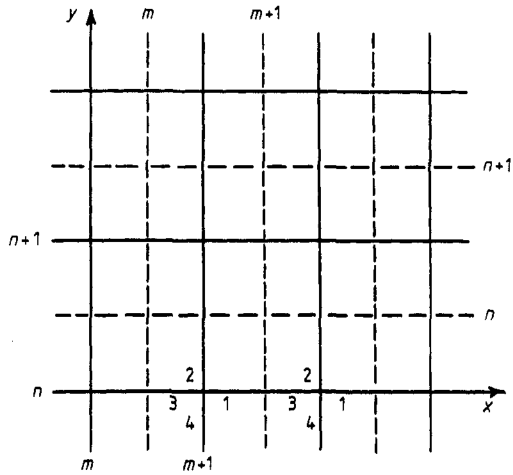

Figure 4. Full lines represent a rectangular Josephson lattice (one Josephson junction on each lattice bond; the rest of the volume is filled by a superconductor). Broken lines represent its dual lattice.

Now we consider rectangular Josephson lattices (one junction on each bond of the 2D lattice). The relation (18) must be satisfied in each site with index (m, n) (see figure 4):

$$\theta_{mn}^{(1)} + \theta_{mn}^{(2)} = \theta_{mn}^{(3)} + \theta_{mn}^{(4)}. \tag{19}$$

The phase difference $\theta(l), l = x, y$, determined on each bond satisfies equation (12). Let us consider the bond between the sites (m, n) and $(m + 1, n)$. The related function $\theta(x)$ is defined as $\theta_{m,n}^{m+1,n}(x)$. A function $\theta_{m,n}^{m,n+1}(y)$ is related to the bond between the sites (m, n) and $(m, n + 1)$. We can introduce the following representation:

$$\begin{aligned} \theta_{m,n}^{m+1,n}(x) &= \varphi_{m,n-1} - \varphi_{m,n} + \tilde{\theta}_n(x) \\ \theta_{m,n}^{m,n+1}(y) &= \varphi_{m,n} - \varphi_{m-1,n} + \tilde{\theta}_m(y) \end{aligned} \tag{20}$$

where $\tilde{\theta}_n(x)$ and $\tilde{\theta}_m(y)$ are continuous functions on the lattice and satisfy equation (12). The phases φ_{mn} are related to the dual lattice represented by the broken lines on figure 4 (the site index (m, n) of the dual lattice is also the index of the elementary plaquette) and may be considered as the phase of the order parameter in superconducting grains. It is easy to prove that the representation (20) satisfies the relation (19). Integrating equation (13) over the bonds around the elementry plaquette (m, n) and using the continuity of the magnetic field $H(l)$ on the bonds, one obtains an important relation:

$$\begin{aligned} &\int_{Y_n}^{Y_{n+1}} \mathrm{d}y\{\sin[\varphi_{mn} - \varphi_{m-1,n} + \tilde{\theta}_m(y)] + \sin[\varphi_{mn} - \varphi_{m+1,n} - \tilde{\theta}_{m+1}(y)]\} \\ &\qquad + \int_{X_m}^{X_{m+1}} \mathrm{d}x\{\sin[\varphi_{mn} - \varphi_{m,n+1} + \tilde{\theta}_{n+1}(x)] \\ &\qquad + \sin[\varphi_{mn} - \varphi_{m,n-1} - \tilde{\theta}_n(x)]\} = 0 \end{aligned} \tag{21}$$

where X_m and Y_m are the x and y components of the radius vector $\boldsymbol{R}_m$ of the site (m, n) of the basic lattice.

Now we study rectangular Josephson lattices with lattice constants $L_x, L_y \gg \lambda$. On each bond at a distance from sites of the lattice, we can assume that the derivative

$\partial\theta/\partial l$ changes slowly over the scale λ (for a proof of the assumption, see the appendix). Therefore, in accordance with (11), the magnetic field $H(l)$ is

$$H(l) = \frac{\phi_0}{4\pi\lambda}\frac{\partial\theta(l)}{\partial l}. \tag{22}$$

Below we will only consider solutions such that $H(l)$ changes slowly near lattice sites. In this case using the continuity of $H(l)$ in each site (m, n) of the basic lattice, we obtain the second condition for the functions $\tilde{\theta}_n(x)$ and $\tilde{\theta}_m(y)$ at the site (m, n):

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_n(x)|_{x=X_m} = \frac{\mathrm{d}}{\mathrm{d}y}\tilde{\theta}_m(y)|_{y=Y_n}. \tag{23}$$

According to equations (20), (22) and (12) the functions $\tilde{\theta}_n(x)$ and $\tilde{\theta}_m(y)$ satisfy the sine–Gordon equation (13). If an external boundary is fixed, then the set of equations (13), (20)–(23) describe completely the behaviour of the system under consideration. Different solutions of the set of equations have been found by us in previous papers (Bryksin *et al* 1989a, b), where we have studied the dependence of the low critical field H^*_{c1}, the pinning energy of vortices and the penetration depth of magnetic fields on the ratio between L_x, L_y and δ at $L_x, L_y \gg \lambda$.

5. Effective-medium approach

In the present section we shall study rectangular Josephson lattices with the lattice constants L_x and L_y whose magnitudes are arbitrary compared to λ. As above we suppose that $H(\boldsymbol{r}_l)$ ($\boldsymbol{r}_l$ is the radius vector of a point on a lattice bond) changes slowly over the scale λ. If $\partial\theta/\partial l$ changes slowly over this scale also, then equation (11) may be written as

$$H(\boldsymbol{r}_l) = \frac{\phi_0}{4\pi\lambda}P(\boldsymbol{r}_l)\frac{\partial\theta(l)}{\partial l} \tag{24}$$

where

$$P(\boldsymbol{r}_l) = \frac{1}{\pi\lambda}\int \mathrm{d}l'\, K_0\left(\frac{|\boldsymbol{r}_l - \boldsymbol{r}_{l'}|}{\lambda}\right). \tag{25}$$

Here the integration is taken over all lattice bonds. Let $\boldsymbol{r}_l$ lie on a bond that is parallel to the x axis. We can divide the integral (25) into two contributions from x and y bonds respectively:

$$\begin{aligned} P(x) &= \frac{1}{\pi\lambda}\left(\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x'\, K_0(\lambda^{-1}[(x-x')^2 + (nL_y)^2]^{1/2})\right. \\ &\qquad \left. + \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}y'\, K_0(\lambda^{-1}[y'^2 + (mL_x - x)^2]^{1/2})\right) \\ &= \coth\frac{L_y}{2\lambda} + \frac{\cosh[(L-2x)/2\lambda]}{\sinh(L_x/2\lambda)}. \end{aligned} \tag{26}$$

In (26) the function $P(x)$ is determined in an interval $0 \leqslant x \leqslant L_x$. Then the function $P(x)$

has to be continued periodically, that is $P(x) = P(x + mL_x)$. If $\boldsymbol{r}_l$ lies on a y bond then the function $P(y)$ has the form (26) with substitutions $L_x \leftrightarrow L_y$ and $x \leftrightarrow y$ ($P(y) = P(y + nL_y)$). In spite of the assumption about a small change of the function $\partial\theta/\partial l$ over the scale λ, we can see from equations (24) and (26) that $P(\boldsymbol{r}_l)$ and consequently $H(\boldsymbol{r}_l)$ does not change slowly over the scale. However, in two limit cases $\lambda \gg L_x, L_y$ and $\lambda \ll L_x, L_y$ this difficulty may be avoided. In the first case ($\lambda \gg L_x, L_y$—the 'dense' lattice case) $P(\boldsymbol{r}_l)$ is a periodic function with a small oscillation amplitude that at $L_x = L_y = L$ is equal to $[\cosh(L/2\lambda) - 1]/\sinh(L/2\lambda) \simeq L/4\lambda \ll 1$. In the second case ($\lambda \ll L_x, L_y$—the 'rare' lattice case) the dependence of $P(\boldsymbol{r}_l)$ on coordinates is important only in small regions with a size of order λ near lattice sites. In these two limit cases $P(\boldsymbol{r}_l)$ may be substituted by its average value $\bar{P}$:

$$\bar{P}_x = \frac{1}{L_x}\int_0^{L_x} \mathrm{d}x\, P(x) \coth\frac{L_y}{2\lambda} + \frac{2\lambda}{L_x}$$
$$\bar{P}_y = \frac{1}{L_y}\int_0^{L_y} \mathrm{d}y\, P(y) = \coth\frac{L_x}{2\lambda} + \frac{2\lambda}{L_y}. \tag{27}$$

Therefore, in the framework of the effective-medium approach, the phase difference $\theta(l)$ obeys the following equation on each bond:

$$\sin\theta(l) = \delta_l^{*2}\frac{\partial^2\theta}{\partial l^2}$$
$$\delta_l^* = \delta\bar{P}_l^{1/2} \tag{28}$$

where $l = x, y$. It is easy to find that in the two limit cases considered above $\bar{P}_l$ and δ_l^* are isotropic ($\bar{P}_x = \bar{P}_y = \bar{P}$, $\delta_x^* = \delta_y^* = \delta^* = \delta\bar{P}^{1/2}$. For $\lambda \gg L_x, L_y$ we have $\bar{P} \simeq 2\lambda(L_x^{-1} + L_y^{-1})$. For $\lambda \ll L_x, L_y$ one obtains $\bar{P} \simeq 1$. If $L_x \gg \lambda$ and $L_y \ll \lambda$ then we have $\bar{P} \simeq 2\lambda/L_y$. For the contrary case $L_x \ll \lambda$ and $L_y \gg \lambda$ we have $\bar{P} \simeq 2\lambda/L_x$. However, equation (28) is incorrect when L_x or L_y is of order of λ. Nevertheless, we believe that the equation gives qualitatively correct results.

In each lattice site the function $\theta(l)$ and its derivative $\partial\theta/\partial l$ are connected by means of equations (19) and (23). Therefore, to describe a Josephson lattice we have to find a self-consistent solution of equation (28) on each bond and then to connect the solution by means of equations (19) and (23). This procedure has been done in our previous papers (Bryksin *et al* 1989a, b) for the case $\lambda \ll L_x, L_y$ when $\bar{P} \simeq 1$. Generalising the procedure for the case of an arbitrary ratio between λ and L_x, L_y, we obtain the following free-energy functional:

$$F = L_xL_y\sum_{m,n}\left\{\left(\frac{\hbar}{2e}j_c\right)\left[\frac{1}{L_x} + \frac{1}{L_y} - \frac{1}{L_x}\cos\left(\varphi_{mn} - \varphi_{m-1,n} - \frac{2\pi}{\phi_0}L_xA_{x,mn}\right)\right.\right.$$
$$\left.\left. - \frac{1}{L_y}\cos\left(\varphi_{mn} - \varphi_{m,n-1} - \frac{2\pi}{\phi_0}L_yA_{y,mn}\right)\right] + \frac{1}{8\pi\mu}h_{mn}^2\right\}. \tag{29}$$

Here

$$\mu = \frac{2\lambda}{\bar{P}}\left(\frac{1}{L_x} + \frac{1}{L_y}\right) \tag{30}$$

is the effective magnetic permeability of the Josephson lattice. The sum in (29) is taken over sites of the dual lattice. The effective magnetic field h is related to the magnetic field H and 'vector potential' $\boldsymbol{A} = (A_x, A_y, 0)$ by means of the following equation:

$$h_{mn} = \mu H_{mn} = \frac{1}{L_x}(A_{y,mn} - A_{y,m-1,n}) - \frac{1}{L_y}(A_{x,mn} - A_{x,m,n-1}) \qquad (31)$$

which in the continuum limit takes the form $\boldsymbol{h} = \operatorname{rot} \boldsymbol{A}$, where $\boldsymbol{h} = (0, 0, h)$. In cases when the phase φ and the 'vector potential' $\boldsymbol{A}$ alter slowly from one site to another, the cos terms in (29) may be expanded in a row. Keeping only terms of the second order of magnitude in the continuum limit we have

$$F = \int \mathrm{d}x\, \mathrm{d}y \left\{ \frac{\hbar}{4e} j_c \left[L_x \left(\frac{\partial \varphi}{\partial x} - \frac{2\pi}{\phi_0} A_x \right)^2 + L_y \left(\frac{\partial \varphi}{\partial y} - \frac{2\pi}{\phi_0} A_y \right)^2 \right] + \frac{1}{8\pi\mu} h^2 \right\}. \qquad (32)$$

In the framework of a phenomenological approach, a similar effective free energy has been used to study properties of granular superconductors (Rosenblatt 1974, Clem and Kogan 1987, Sonin 1988).

The functional (29) can be applied to the description of Josephson lattices if the magnetic field changes slowly along a bond. The condition does not take place when either L_x or L_y is larger than δ or of the same order as δ. In the previous papers (Bryksin *et al* 1989a, b) we have shown that highly anisotropic Josephson lattices (either $L_x \ll \delta \ll L_y$ or $L_y \ll \delta \ll L_x$) are described by the Frenkel–Kontorova model (Bak and Bohr 1982).

Let us discuss the penetration of a low magnetic field into Josephson lattices described by the functional (29). It is easy to find that for square lattices ($L_x = L_y = L$) the effective penetration depth is equal to

$$\lambda^* = \delta \left(\frac{2\lambda}{L\mu} \right)^{1/2}. \qquad (33)$$

At $\lambda \ll L \ll \delta$ in accordance with (27) and (30) we have $\mu \simeq 4\lambda/L$ and consequently $\lambda^* \simeq \delta/\sqrt{2}$ (Sonin 1988, Bryksin *et al* 1989a, b). In the case $L \ll \lambda$ one obtains $\mu \simeq 1$ and $\lambda^* \simeq \delta(2\lambda/L)^{1/2} \gg \delta$. In the general case according to the results of Sonin (1988) and Bryksin *et al* (1989a, b) the low critical magnetic field (H^*_{c1}) above which vortices arise is equal to

$$H^*_{c1} = \frac{\phi_0}{4\pi\mu\lambda^{*2}} \left(\ln \frac{\lambda^*}{L} + \text{constant} \right). \qquad (34)$$

Let us compare H^*_{c1} and the low critical field $H^{(A)}_{c1}$ of Abrikosov vortices:

$$H^{(A)}_{c1} = \frac{\phi_0}{4\pi\lambda^2} \left(\ln \frac{\lambda}{\xi} + \text{constant} \right) \qquad (35)$$

where ξ is the correlation length. For $\lambda, L \ll \delta$, taking into account (33), (34) and (35), we have

$$\frac{H^{(A)}_{c1}}{H^*_{c1}} = \frac{\delta^2 \ln(\lambda/\xi)}{\lambda L \ln(\lambda^*/L)} \gg 1 \qquad (36)$$

that is $H^{(A)}_{c1}$ is much larger than H^*_{c1}. Vortices in a Josephson lattice are pinned to sites of the lattice. At $\lambda \ll L \ll \delta$ the energy of pinning is equal to

$$E_p = 0.1(\hbar/e) j_c L \qquad (37)$$

(Lobb *et al* 1983, Bryksin *et al* 1989a, b). The result is also correct at $\lambda \gg L$.

Finally, we study the case of high external magnetic fields H_0 ($H_{c1}^* \ll H_0 \ll H_{c1}^{(A)}$) when the field H determined by (24) is approximately equal to H_0. The case $\lambda \ll L$ has been studied by us (Bryksin *et al* 1989a, b). Here we generalise these results and consider the case $\lambda \gg L$ also.

According to (24) where $P(r_l)$ is substituted by $\bar{P}$ the phase difference $\theta(l)$ on each bond is a linear function of the coordinate $l = x, y$:

$$\theta(l) = \theta + \frac{4\pi\lambda H_0}{\phi_0 \bar{P}} l. \tag{38}$$

In the case under consideration the energy functional may be chosen in the form

$$F = -\frac{\hbar}{2e} j_c \sum_{\text{bond}} \int_{\text{bond}} \mathrm{d}l \cos\theta(l) \tag{39}$$

where the sum is taken over all lattice bonds. Inserting (38) into (39) and then using the transformation (20) to the dual lattice, we can write the following energy functional of a square Josephson lattice in high magnetic fields:

$$F = -\frac{\hbar j_c \phi_0 \bar{P}}{8\pi e \lambda H_0} \sin\left(\frac{2\pi\lambda L H_0}{\phi_0 \bar{P}}\right) \sum_{i,g} \cos\left(\varphi_{i+g} - \varphi_i - \frac{8\pi\lambda}{\phi_0 \bar{P}} (\boldsymbol{A}_{i,g}, \boldsymbol{g})\right) \tag{40}$$

where the sum is taken over all sites of the dual lattice $i = (m, n)$; $\boldsymbol{g}$ is the unit vector of a nearest neighbour. The vector potential $\boldsymbol{A}_{i,g}$ is determined by

$$\boldsymbol{A}_{i,g} = \tfrac{1}{4}[\boldsymbol{H}_0 \times (\boldsymbol{R}_i + \boldsymbol{R}_{i+g})] \tag{41}$$

where $\boldsymbol{R}_i$ is a radius vector of the site i.

For $L \gg \lambda$ according to (27) we have $\bar{P} \simeq 1$. In this case the functional (40) shows that in the region of high magnetic fields $\phi_0/\lambda L < H_0 \ll H_{c1}^{(A)}$ the coupling strength between superconducting grains depends strongly on H_0 and is proportional to $(\phi_0/\phi) \sin(2\pi\phi/\phi_0)$, where $\phi = \lambda L H_0$ is half of the magnetic field flux through a bond (junction).

For a lattice with $L \ll \lambda$ we have $\bar{P} \simeq 4\lambda/L$. In this case the dependence of the coupling strength on H_0 may arise only at $H_0 > \phi_0/L^2 \gg H_{c1}^{(A)}$. In the region $H_0 < H_{c1}^{(A)}$ the functional (40) takes the form

$$F = -\frac{\hbar j_c L}{4e} \sum_{i,g} \cos\left(\varphi_{i+g} - \varphi_i - \frac{2\pi L}{\phi_0} (\boldsymbol{A}_{i,g}, \boldsymbol{g})\right). \tag{42}$$

It is this functional that is usually used to study Josephson lattices in magnetic fields.

Let us give rough estimates of the critical field H_{c1}^*. For typical high-T_c superconductors, the low critical field $H_{c1}^{(A)}$ is of order 10^2 Oe, $\lambda \sim 0.1$ μm and the grain size $L \sim 1$ μm. The Josephson length varies from 1 to 10^2 μm depending on the barrier thickness d. Using (6) we obtain that H_{c1}^* lies in the range 10–10^{-3} Oe. These values of H_{c1}^* are in agreement with the data of Kwak *et al* (1988) and Mazaki *et al* (1987), for example. If L is smaller and δ is larger, then H_{c1}^* may be lower.

6. Conclusions

In the present paper we have applied a general description of a Josephson junction with complex form to investigate properties of Josephson lattices. Although we limit ourselves to rectangular lattices, we think that the approach proposed above may also be applied to a disordered lattice.

Above we have studied only stationary properties. However, it is evident that our approach may be generalised to study non-equilibrium phenomena. The first step in this direction has been done in our previous papers (Bryksin *et al* 1989a, b), where we have considered oscillations of a vortex about a pinning centre.

Since magnetic fields in superconducting grains are described by equation (3), our results are correct only for an external field H_0 smaller than the low critical magnetic field $H_{c1}^{(A)}$ of Abrikosov vortices. At $H_0 > H_{c1}^{(A)}$ Abrikosov vortices penetrate inside the grains. In this case we must put the sum of point sources of magnetic fields (that is $\Sigma_i \phi_0 \delta(\boldsymbol{r} - \boldsymbol{r}_i)$, where $\boldsymbol{r}_i$ is the radius vector of the vortex centre) in the right-hand side of equation (3). A self-consistent solution of equations (1) and (2) and the modified equation (3) allows us to study the interaction between Abrikosov vortices and Josephson junctions.

Appendix

Let us consider magnetic field penetration in a semi-infinite ($x \geqslant 0$) linear Josephson junction. We will show that both magnetic field and phase difference θ change slowly everywhere over the scale λ including a region ($0 \leqslant x \leqslant \lambda$) near the junction boundary.

At low magnetic fields equation (12) may be linearised and takes the form

$$\theta(x) = \frac{c}{4\pi j_c}\frac{\mathrm{d}H}{\mathrm{d}x} \tag{A1}$$

where $H(x)$ is determined by equation (5). The function $\theta(x)$ and 'surface charge' $\sigma(x)$ are related by equation (10). Inserting (10) and (A1) into (5) we have

$$H(x) = \frac{\delta^2}{\pi\lambda}\int_0^\infty \mathrm{d}x' \frac{\mathrm{d}^2H}{\mathrm{d}x'^2}\left[K_0\left(\frac{|x-x'|}{\lambda}\right) - K_0\left(\frac{|x+x'|}{\lambda}\right)\right] + H_0\,\mathrm{e}^{-x/\lambda}. \tag{A2}$$

Now we define the function $H(x)$ at negative x:

$$H(x) = -H(-x). \tag{A3}$$

Then (A2) may be written as

$$H(x) = \frac{\delta^2}{\pi\lambda}\int_{-\infty}^\infty \mathrm{d}x' \frac{\mathrm{d}^2H}{\mathrm{d}x'^2} K_0\left(\frac{|x-x'|}{\lambda}\right) + \mathrm{sgn}(x)H_0\,\mathrm{e}^{-|x|/\lambda}. \tag{A4}$$

This equation may be easily solved by means of Fourier transformation. The solution of (A4) is

$$H(x) = \frac{H_0}{2\pi}\int_{-\infty}^\infty \mathrm{d}k\, 2\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}kx}\left(\frac{1}{\lambda^{-2}+k^2} + \frac{\delta^2}{\lambda(\lambda^{-2}+k^2)^{1/2}}\right) \times \left(1 + \frac{\delta^2k^2}{\lambda(\lambda^{-2}+k^2)^{1/2}}\right)^{-1}. \tag{A5}$$

At $x > 0$ in a plane of complex k, contributions to the integral (A5) are determined by a pole at $k = -\mathrm{i}\kappa/\lambda$, where

$$\kappa = \frac{\lambda}{\delta}\left[\left(1+\frac{\lambda^2}{4\delta^2}\right)^{1/2} - \frac{\lambda^2}{2\delta^2}\right]^{1/2} \tag{A6}$$

and a cut along the imaginary axis from $-\mathrm{i}/\lambda$ to $-\mathrm{i}\infty$. As a result of calculations, equation (A5) takes the form

$$\frac{H(x)}{H_0} = \frac{2}{2-\kappa^2}\,\mathrm{e}^{-\kappa x/\lambda} - \frac{2\lambda^2}{\pi\delta^2}\int_1^\infty \mathrm{d}u\,\frac{u\,\mathrm{e}^{-ux/\lambda}}{(u^2-1)^{1/2}(u^2-\kappa^2)(u^2+\gamma^2)} \tag{A7}$$

where $\gamma^2 = \kappa^2(1-\kappa^2)^{-1}$. The first term in (A7) is the contribution of the pole. The second term is the contribution of the cut. The first term changes slowly over the scale λ since at $\lambda/\delta \ll 1$ we have $\kappa \simeq \lambda/\delta$. The second term decreases exponentially in a distance of the order of λ from the boundary. However, because of the small multiplier $\lambda^2/\delta^2 \ll 1$ the second term in (A7) is small at all x. Therefore we can consider that at all x the magnetic field $H(x)$ determined by (A7) changes slowly over the scale λ. The derivative $\mathrm{d}H/\mathrm{d}x$ and $\theta(x)$ (see (A1)) also possess the same property. It is useful to note that at $x < \lambda$ the contribution of the second term to $\mathrm{d}^2H/\mathrm{d}x^2$ is of the same order of magnitude as the first term. Moreover, at $x = 0$ the derivative $\mathrm{d}^2H/\mathrm{d}x^2$ and consequently $\mathrm{d}\theta/\mathrm{d}x$ are equal to zero. At $x \to 0$ all high derivatives tend to infinity. However at distances $x > \lambda$ the function $\mathrm{d}\theta/\mathrm{d}x$ changes slowly again. The fact approves the use of equation (22).

Note added in proof. After we had submitted this paper, two papers (Volkov 1989a, b) were published where for the system of parallel Josephson junctions the equation (14) was also obtained.

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